UK Junior Mathematical Olympiad 2001

Organised by The United Kingdom Mathematics Trust

Tuesday 12th June 2001

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators *and measuring instruments* is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required.* Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first hour so as to allow at least one hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Numerical answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

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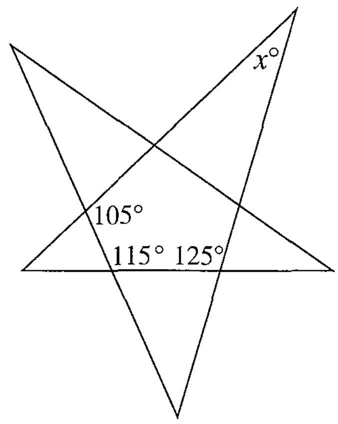
Section A

A1 In the late 18th century, a decimal clock was proposed, in which there were 100 'minutes' in one 'hour' and 10 'hours' in one day. Assuming that such a clock started from 0:00 at midnight, what time would it show when an ordinary clock showed 6 o'clock the following morning?

A2 $\square + 8 \div 2 = 10$.
Which number should replace the box to make a true statement?



A3 What is the value of x in the diagram alongside?

A4 The areas of three of the faces of a cuboid are 24 cm^2, 18 cm^2 and 12 cm^2. What is the volume of the cuboid ?

A5 Find the 100th digit after the decimal point in the decimal representation of $\frac{3}{7}$.

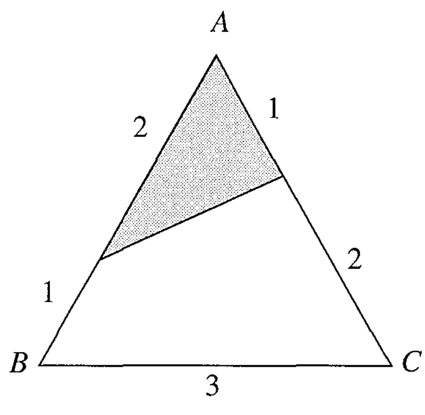
A6 'One third of the population now has access to the internet; this is 50% more than one year ago.'
What fraction of the population had access to the internet one year ago?

A7 The length of each side of a quadrilateral *ABCD* is a whole number of centimetres. Given that $AB = 4$ cm, $BC = 5$ cm and $CD = 6$ cm, what is the maximum possible length of the fourth side *DA*?

A8 Find the smallest three-digit number which is neither prime nor divisible by 2, 3 or 5.

A9 Points *A*, *B*, *P*, *C*, *Q*, *D*, *R*, *E*, *S* and *F* are marked in that order around the circumference of a circle so that *ABCDEF* is a regular hexagon and *APQRS* is a regular pentagon.
What is the size of $\angle BAP$?

A10 What fraction of triangle *ABC* is shaded?

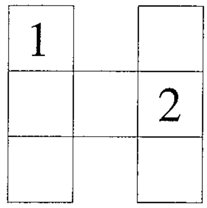


Section B

Your solutions to Section B will have a major effect on the JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief 'answers').

B1 The numbers from 1 to 7 inclusive are to be placed, one per square, in the diagram on the right so that the totals of the three numbers in the horizontal row and each of the two columns are the same.

In how many different ways can this be done if the numbers 1 and 2 must be in the positions shown?



B2 In a sequence, each term after the first is the sum of the squares of the digits of the previous term. Thus, if the first term were 12, the second term would be $1^2 + 2^2 = 5$, the third term $5^2 = 25$, the fourth term $2^2 + 5^2 = 29$ and so on.

(i) Find the first five terms of the sequence whose first term is 25.

(ii) Find the 2001st term of the sequence whose first term is 25.

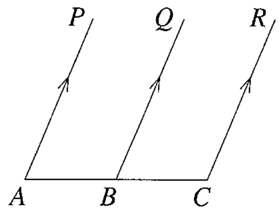
B3 In the diagram, B is the midpoint of AC and the lines AP, BQ and CR are parallel. The bisector of $\angle PAB$ meets BQ at Z.

Draw a diagram to show this, and join Z to C.

(i) Given that $\angle PAZ = x°$, find $\angle ZBC$ in terms of x.

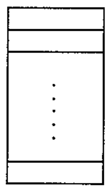
(ii) Show that CZ bisects $\angle BCR$.

(You must give full reasons to justify your answers.)



B4 The diagram shows a large rectangle whose perimeter is 300 cm. It is divided up as shown into a number of identical rectangles, each of perimeter 58 cm. Each side of these rectangles is a whole number of centimetres.

Show that there are exactly two possibilities for the number of smaller rectangles and find the size of the large rectangle in each case.

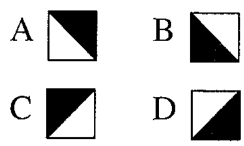


B5 Observe that $49 = 4 \times 9 + 4 + 9$.

(i) Find all other two-digit numbers which are equal to the product of their digits plus the sum of their digits.

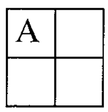
(ii) Prove that there are no three-digit numbers which are equal to the product of their digits plus the sum of their digits.

B6 This question is about ways of placing square tiles on a square grid, all the squares being the same size. Each tile is divided by a diagonal into two regions, one black and one white. Such a tile can be placed on the grid in one of four different positions as shown:



When two tiles meet along an edge (side by side or one below the other) the two regions which touch must be of different types (i.e. one black and one white).

(i) A 2×2 grid of four squares is to be covered by four tiles.



(a) If the top-left square is covered by a tile in position A, find all the possible ways in which the other three squares may be covered.

(b) In how many different ways can a 2×2 grid be covered by four tiles?

(ii) In how many different ways can a 3×3 grid be covered by nine tiles?

(iii) Explaining your reasoning, find a formula for the number of different ways in which a square grid measuring $n \times n$ can be covered by n^2 tiles.